Multifractals of Normalized First Passage Time in Sierpinski Gasket

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The multifractal behavior of the normalized first passage time is investigated on the two dimensional Sierpinski gasket with both absorbing and reflecting barriers. The normalized first passage time for Sinai model and the logistic model to arrive at the absorbing barrier after starting from an arbitrary site, especially obtained by the calculation via the Monte Carlo simulation, is discussed numerically. The generalized dimension and the spectrum are also estimated from the distribution of the normalized first passage time, and compared with the results on the finitely square lattice.

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1 Introduction

Recently a lot of interest has been concentrated on the behavior of the disorder system in a variety of contexts in condensed matter physics[1-5]. In the transport process one of the important subjects for the regular and disorder systems is the random walk theory, and this theory has extensively been developed to the continuous-time random walk theory which is essentially described both by the transition probability dependent of the length between steps and by the distribution of the pausing times [5-8]. Until quite recent the transport phenomena for the motion of a particle have been largely extended to the reaction kinetics [5,9], and the strange kinetics[10].

One of the well-known interesting issues in the theory of the random walk is the mean first passage time that is defined by the average time arriving at the absorbing barrier for first time after a particle initially starts from an arbitrary site. In particular, this problem has extensively been studied in the Sinai model which is discussed on the random barrier model such as the absorbing and reflecting barriers. In previous work the Sinai model with asymmetric transition probability also was studied for the mean and mean square displacements dependent anomalously on time [2-7]. Moreover, the transport process for the mean first passage time has already been addressed by Noskowicz et al.[11] who have obtained the upper and lower bounds using the recursion - relation procedure [12-15]. In one dimensional lattice with the periodic boundary condition, Kozak et al.[16] treated long ago with the first passage time of a particle for the case of the transition probability as a chaotic orbits of logistic map [17], and obtained the statistical value of the mean first passage time with the anomalous exponent.

On the other hand, since the multifractal characteristics is directly related to analyze the statistical property of the normalized first passage time, the multifractal for the mean first passage time has been studied intensively in connection with the random walk problem in the one dimensional lattice with both absorbing and

reflecting barriers[15,18]. In general, the multifractal investigation for the chaotic and disorder systems have been applied analytically and numerically to the transport phenomena [19,20] and the random fractal structure having the time-dependent random potential in the diffusive motion[21-24]. Murphy et al[15] recently has argued on the multifractal behavior for the mean first passage time in one dimensional lattice having both absorbing and reflecting barriers. Therefore, we think that it is of great interest for the multifractal investigation to treat with the transport process described by the transition probability such as logistic map.

The purpose of this paper is to investigate the multifractal behavior of a particle executed on the random walk in the two dimensional Sierpinski gasket. Specifically, we deal with the random walk for one particle having the transition probability given by the logistic map, and concentrate on the normalized first passage time for the random walk with symmetric and asymmetric transition probabilities on Sierpinski gasket existing both absorbing and reflecting barriers. In section 2, we present the transition probability expressed in terms of the logistic map, and the convenient formulas of the normalized first passage time and the multifractal are introduced. In section 3, the distribution of normalized first passage time arriving at an absorbing barrier is considered in both Sinai model and the logistic model which has the transition probability given by logistic map. For simplicity, the generalized dimension and the spectrum from the statistical value of normalized first passage time are also calculated numerically, and compared with the results reported earilier [18]. Lastly, we give some conclusions.

2 Normalized first passage time and multifractals

First of all,we present the transition probability as a chaotic orbits of logistic map in this section. We also introduce the formulas of

the normalized first passage time, the generalized dimension, and the spectrum.

Next, we shall consider the random walk of a particle in two-dimensional Sierpinski gasket in which the number of sites N_n having stage n in the d-dimensional Sierpinski gasket[25] is given by $N_n = (d+1)(1+(d+1)^n)/2$. It is assumed that a particle is started from an arbitrary site A on Sierpinski gasket. The reflecting barriers are located on all the boundary, except that an absorbing barrier is at an ended site B under the right side, as shown in Fig.1.

Now, the transition probability for the motion of a particle is introduced as

$$x(n+1) = Rx(n)[1 - x(n)] \tag{1}$$

where R is the controll parameter with $0 < R < 4$. In Monte Carlo simulation one has to treat with the arbitrary values of R, where the numerical generated sequences appear to be chatic[20,26]. By using the transition probability as Eq. 1, one particle jumps to right site with $p_{1j} + \gamma$ or left site with $q_{1j} - \gamma$, and to up site with $p_{2j} + \gamma$ or down site with $q_{2j} - \gamma$ in four directions after a particle starts from a site j on two-dimensional Sierpinski gasket, where γ is called the disorder parameter. It has really been showed that the normalized condition for transition probability is given by $p_{1j} + q_{1j} + p_{2j} + q_{2j} = 1$. In particular, the transition probability is symmetric for the disorder parameter $\gamma = 0$, and for $0 < \gamma < \frac{1}{4}$, it also assumes that a particle executes on a biased random walk jumped to the direction of the absorbing barrier.

Using the generating function technique the mean first passage time[2,3] $< T >$ in one dimensional lattice is given by

$$< T > = \sum_{k=1}^{N-1} \frac{1}{p_k} + \sum_{k=0}^{N-2} \frac{1}{p_k} \sum_{i=k+1}^{N-1} \prod_{j=k+1}^{i} \frac{q_j}{p_j}. \tag{2}$$

where one particle moves to right site with p_j or left site with q_j after one step, when it starts from site j. However, the transition probability has always the different value at each site when a particle jumps for ocurrence of the one step to a nearest-neighbor site from a given site, and the derivation for Eq. 2 can be found in details elsewhere[11,12,15]. Particularly, a possible extension of this paper is only to two dimension in order to discuss briefly on the mean first passage time and the multifractal in two dimensional Sierpinski gasket. The normalized first passage time T_{nt} has the form

$$T_{nt} = \frac{t_n - t_{min}}{t_{max} - t_{min}} \tag{3}$$

where t_n is the first passage time arrivng at the absorbing barrier after n steps, and t_{max} and t_{min} are the maximum and minimum values of the first passage time arrivng at the absorbing barrier, respectively. In the next section we will perform the conventional numerical simulations for the normalized first passage time in both Sinai model and the logistic model on two-dimensional Sierpinski gasket.

To show the multifractal feature for the motion of a particle with the transition probability expressed in terms of the logistic function, we extend to the multifractal calculations for the normalized first passage time. If we divide the normalized first passage time into a set ϵ of as $\epsilon \to 0$, then the generalized dimension in the multifractal structure[19,20] is represented as

$$D_q = \lim_{\epsilon \to 0} \frac{ln \sum_i n_i {p^q}_i}{(q-1) ln\epsilon} \tag{4}$$

where p_i is the probability for the number of configuration of i^{th} particle arrived at the absorbing barrier and n_i the number of configuration for i^{th} particle. By introducing the above expression the

spectrum f_q and α_q is caculated from the relations

$$f_q = \alpha_q - (q-1)D_q \tag{5}$$

and

$$\alpha_q = \frac{d}{dq}[(q-1)D_q], \tag{6}$$

where Eq. 5 and Eq. 6 can be obtained by Legendre transformation.

3 Calculation and results

We mainly concentrate on the generalized dimension and the spectrum for the normalized first passage time on two-dimensional Sierpinski gasket lattice. We only restrict ourselves to the case of $n = 7$ stages on Sierpinski gasket, even though the stage n can be extended to large number in this paper. At first, three sites are at $(1), (2)$, and (3) in the initial stage $n = 0$, and we assume that a particle starts from an initial point $i_0 = (11111233)$ at $n = 7$ stages on two-dimensional Sierpinski gasket lattice. The reflecting barriers is located on all the boundary surface, except that an absorbing barrier is at $B = (33333333)$ in Fig.1. Next, both the cases of Sinai model and the logistic model, for asymmetric as well as for symmetric transition probability, are considered as follows: One is the case of Sinai model in which it has the symmetric transition probability with $p_{1j} = q_{1j} = p_{2j} = q_{2j} = \frac{1}{4}$ and $\gamma = 0$, and the two asymmetric transition probabilities with $\gamma = 0.05$ and $\gamma = 0.1$. The other is the case of the logistic model which is related to the transition probability given by the logistic map under the same condition of Sinai model.

From now on, we estimate numerically the generalized dimension and the spectrum after finding the normalized first passage time from Eq. 3 via Monte Carlo simulation. To discuss on these multifractal quantities for the normalized first passage time in two

dimensional Sierpinski gasket, we perform the numerical simulations to show our analytic arguments. For three values of the disorder parameter (i.e., $\gamma = 0, 0.05, 0.1$) in both Sinai model and the logistic model, our simulations are performed 3×10^6 particles and averaged over 10^4 configurations, and analyzed the normalized first passage time, the generalized dimension, and the spectrum. The result of these calculations is summarized in Table 1.

Particularly, in the logistic model we are actually interested in the one case for the control parameter $R = 3.9999$. It can be easily found from the iteration of the logistic map that three symmetric cutoff values are 0.152210, 0.505753, and 0.855890 for $\gamma = 0$, while three asymmetric cutoff values are 0.280024, 0.505753, and 0.730526 for $\gamma = 0.1$, respectively. Here the cutoff value is defined as the quantity used to determine the direction of random walker in the logistic model.

As shown in Table 1, the fractal dimenion D_o(i.e., the maximum value of D_q or f_q), the scaling exponent α_q, and the generalized dimension D_q, ultimately based on the theoretical expressions Eq. 4 - Eq. 6, are calculated numerically in both Sinai model and the logistic model. In Sinai model Figs. 2 and 3 depict respectively the generalized dimension D_q as a function of q and the spectrum f_q as a function of α_q for varying the disorder parameter values. It can be seen from Table 1 that the fractal dimension changes anomalously as the disorder parameter γ increases in two models. In particular, it is also found from the result obtained in our simulations that in Sinai model for $\gamma = 0.05$ the value of the fractal dimension on Sierpinski gasket is nearly equal to that on square lattice. Futhermore, in the logistic model on the two dimensional Sierpinski gasket, the normalized first passage time is found to be infinite for $\gamma = 0$, and the fractal dimension is expected to take the value near zero as $\gamma \to 0.5$.

In conclusion, we have here considered the normalized first passage time for both Sinai model and the logistic model on two dimen-

sional Sierpinski gasket. We have clearly showed the multifractal characteristics from the distribution of the normalized first passage time, as summarized in Table 1. In future work, for the multifractal characteristics for the normalized first passage time, we intend to compare with the results found in both Sinai model and the logistic model if the transition probability with the modified log - normal function is introduced, and to attempt to investigate extensively in the similar lattice models.

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Figures

Fig.1. The trajectory of a particle in the two dimensional Sierpinski gasket at $n = 7$ stages. A particle is started from at $A = (11111233)$, and the absorbing barrier represents the site $B = (33333333)$.

Fig.2. The generalized dimension D_q versus q for the normalized first passage time on the two dimensional Sierpinski gasket in Sinai model.

Fig.3. The spectrum f_q versus α_q for the normalized first passage time on the two dimensional Sierpinski gasket in Sinai model.

Table

Table I. Values of the fractal dimension, the generalized dimension, and the scaling exponent in Sinai model and the logistic model. These lattice models are on both Sierpinski gasket and the finite square lattice[18].